

A Multiplication Rule for the Descent Algebra of Type D

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Abstract

Here we give an interpretation of Solomon's rule for multiplication in the descent algebra of Coxeter groups of type D , ΣD_n . We describe an ideal \mathcal{I} such that $\Sigma D_n/\mathcal{I}$ is isomorphic to the descent algebra of the hyperoctahedral group, ΣB_{n-2} .

1 Introduction

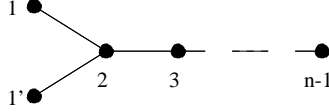
Given a Coxeter group, W , we can construct an algebra - *the descent algebra* - which is a sub-algebra of the group algebra $\mathbb{Q}[W]$. These were introduced in 1976 by Louis Solomon [7]. A revival of interest in this area began in the 80's when applications were found for an interpretation of the rule for multiplying together basis elements of the descent algebra of the symmetric group, for example [6], [5]. Since this interpretation involved matrices, we shall call it the "matrix interpretation" from here on. This matrix interpretation provided the key to many advances in the subject (for instance [1], [4], [3]) including an analogous matrix interpretation by François and Nantel Bergeron for the descent algebra of the hyperoctahedral group, [2].

Until now, there has been little success in developing such an interpretation for the Coxeter groups of type D . However, in this paper we shall give the matrix interpretation for this remaining Coxeter family, after defining the Coxeter groups of type D , and their corresponding descent algebra.

The n -th Coxeter group of type D , D_n , is the group acting on the set

$$\{-n, \dots, -1, 1, \dots, n\}$$

whose Coxeter generators are the set $S = \{s_{1'}, s_1, s_2, \dots, s_{n-1}\}$, where s_i is the product of transpositions $(-i-1, -i)(i, i+1)$ for $i = 1, 2, \dots, n-1$, and $s_{1'}$ is the product of transpositions $(-2, 1)(-1, 2)$. The relations are given by the following diagram:



where an edge between distinct nodes i and j gives us the relation $(s_i s_j)^3 = 1$, and no edge gives $(s_i s_j)^2 = 1$, and $(s_i)^2 = 1$.

Solomon proved that if J is a subset of S , W_J is the subgroup generated by J , $X_J (X_J^{-1})$ is the unique set of minimal length left (right) coset representatives of W_J , and \mathcal{X}_J is the formal sum of the elements in X_J then for $J, K, L \subseteq S$

$$\mathcal{X}_J \mathcal{X}_K = \sum_L a_{JKL} \mathcal{X}_L$$

where a_{JKL} is the number of elements $x \in X_J^{-1} \cap X_K$ such that $x^{-1} J x \cap K = L$. Hence, the set of all \mathcal{X}_J 's form a basis for an algebra - the descent algebra of $D_n, \Sigma D_n$. Our interpretation of this multiplication rule uses this basis, but for ease of computation, we use a different notation.

We define a composition, q , of an integer, n , to be an ordered list $[q_1, q_2, \dots, q_k]$ of positive integers whose sum is n , and shall write $q \models n$ to denote this. We shall call the integers q_1, q_2, \dots, q_k the *components* of q .

There exists a natural bijection between the subsets of S and the disjoint union, $C(n)$, of the sets $C_{<n} = \{q | q \models m, m \leq n-2\}$, $C_1 = \{q | q \models n, q_1 = 1\}$, $C_n = \{q | q \models n, q_1 \geq 2\}$ and $C'_n = \{q | q \models n, q_1 \geq 2\}$. Note that C_n and C'_n are two copies of the same set. Let $q \in C(n)$ such that $q \models m \leq n$, then the subset corresponding to q is

1. $\{s_{q_0}, s_{q_0+q_1}, \dots, s_{q_0+\dots+q_{(k-1)}}\}$ if $q \in C_{<n}$,
2. $\{s_{1'}, s_1, s_{1+q_2}, \dots, s_{1+q_2+\dots+q_{(k-1)}}\}$ if $q \in C_1$,
3. $\{s_{1'}, s_{q_1}, \dots, s_{q_1+\dots+q_{(k-1)}}\}$ if $q \in C_n$,
4. $\{s_1, s_{q_1}, \dots, s_{q_1+\dots+q_{(k-1)}}\}$ if $q \in C'_n$,

where $q_0 = n - m$.

Remark The step of corresponding a set, J , containing $s_{1'}$ (s_1) with a composition, q , in C_n (C'_n) is because we shall later relate q to the complement of J .

2 The Matrix Interpretation, and Results

If J^c is the complement of J in S , then we let $B_q = \mathcal{X}_{J^c}$ where q is the composition in $C(n)$ that corresponds to J by the above bijection. The matrix interpretation of Solomon's multiplication rule can now be described as follows.

Consider the template with the following form

$$\begin{pmatrix} z_{00} & z_{01} & z_{02} & \dots & z_{0v} \\ & y_{11} & y_{12} & \dots & y_{1v} \\ z_{10} & z_{11} & z_{12} & \dots & z_{1v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & y_{u1} & y_{u2} & \dots & y_{uv} \\ z_{u0} & z_{u1} & z_{u2} & \dots & z_{uv} \end{pmatrix}$$

Note that the y -lines do not have entries in column 0. We say a template is a “filled template” if all entries in a template are non-negative integers.

Definition 1 *Let \mathbf{t} be a filled template. We define the border-sum, $\mathcal{B}(\mathbf{t})$, of \mathbf{t} to be the sum*

$$z_{00} + \sum_{i=1}^u z_{i0} + \sum_{j=1}^v z_{0j}$$

and the y -sum, $\mathcal{Y}(\mathbf{t})$, to be $\sum_{i,j} y_{ij}$. The reading word of \mathbf{t} , $r(\mathbf{t})$, is given by

$$[z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted, unless $z_{00} = 1$, in which case $r(\mathbf{t})$ is given by

$$[1, z_{01}, z_{02}, \dots, z_{0v}, y_{1v}, \dots, y_{12}, y_{11}, z_{10}, z_{11}, z_{12}, \dots, z_{1v}, \dots, z_{u0}, z_{u1}, z_{u2}, \dots, z_{uv}]$$

with zero entries omitted.

If p , and q are compositions in $C(n)$ such that $p \models l \leq n$, and $q \models m \leq n$, then let $Z(p, q)$ be the set of filled templates, \mathbf{t} , such that

1. $z_{0j} + \sum_{i \neq 0} (y_{ij} + z_{ij}) = p_j$, $j \neq 0$,
2. $\sum_i z_{i0} = n - l$,
3. $z_{i0} + \sum_{j \neq 0} (y_{ij} + z_{ij}) = q_i$, $i \neq 0$,
4. $\sum_j z_{0j} = n - m$,
5. If $\mathcal{B}(\mathbf{t})=0$, $\mathcal{Y}(\mathbf{t})$ is odd if
 - (a) $p \in C_1 \cup C_n$ and $q \in C'_n$, or
 - (b) $p \in C'_n$ and $q \in C_1 \cup C_n$.

Otherwise $\mathcal{Y}(\mathbf{t})$ is even.

We are now ready to state our matrix interpretation. To distinguish between those compositions belonging to C_n and those belonging to C'_n , we shall write q' when $q \models n$ and $q \in C'_n$.

Theorem 1 *Let $p, q \in C(n)$. For any filled template \mathbf{t} , let $r(\mathbf{t}) = [r_1(\mathbf{t}), r_2(\mathbf{t}), \dots]$. Then,*

$$B_p B_q = \sum_{\mathbf{t} \in Z(p, q)} \tilde{B}_{r(\mathbf{t})}$$

where $\tilde{B}_{r(\mathbf{t})}$ satisfies the following.

1. If $q \in C_1$, then $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$.
2. If $q \in C_n$, then $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$.
3. If $q \in C'_n$, then if $r_1(\mathbf{t}) = 1$ then $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$, otherwise $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}'$.
4. If $q \in C_{<n}$, then
 - (a) If $r_1(\mathbf{t}) \geq 2$, $p \in C_1 \cup C_n$ and $\mathcal{Y}(\mathbf{t})$ is odd, or $r_1(\mathbf{t}) \geq 2$, $p \in C'_n$ and $\mathcal{Y}(\mathbf{t})$ is even, then $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}'$.
 - (b) If $p \in C_{<n}$ and $z_{00} = 0$, then if $r_1(\mathbf{t}) = 1$, then $\tilde{B}_{r(\mathbf{t})} = 2B_{r(\mathbf{t})}$, otherwise $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})} + B_{r(\mathbf{t})}'$.
 - (c) Otherwise $\tilde{B}_{r(\mathbf{t})} = B_{r(\mathbf{t})}$.

A rigorous proof of this theorem can be obtained through a variety of methods. One is to use shuffle products in a way similar to that seen in [5], or sketched in [2], to prove the analogous interpretations for the descent algebras of the Coxeter groups of types A and B , respectively. Alternatively, this theorem can be proved using the general framework suggested in [8]. Indeed this framework inspired Theorem 1, and a proof in this vein can be found in [9].

Here, however, we wish to emphasize that it is the formulation of Theorem 1 that is the most difficult stage. Once this has been achieved, a proof can be derived by the diligent reader, with or without the use of the above references, or found in [9]. Therefore we feel it would be more beneficial to replace the proof with a collection of illuminating examples.

Examples To illustrate our rule we shall work in ΣD_4 . Each example, $B_p B_q$, shall consist of $Z(p, q)$, and the resulting summands it generates according to the rule.

$$1. B_{[4]}B_{[1,3]}.$$

$$\begin{pmatrix} 0 & 0 \\ & 1 \\ 0 & 0 \\ & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ & 0 \\ 0 & 1 \\ & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ & 0 \\ 0 & 1 \\ & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ & 1 \\ 0 & 0 \\ & 1 \\ 0 & 2 \end{pmatrix}$$

$$B_{[4]}B_{[1,3]} = 2B_{[1,3]} + B_{[1,2,1]} + B_{[1,1,2]}$$

$$2. B_{[3,1]'}B_{[4]}.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$B_{[3,1]'}B_{[4]} = B_{[3,1]} + B_{[1,3]} + 2B_{[1,2,1]}$$

$$3. B_{[2,2]'}B_{[4]'}.$$

$$\begin{pmatrix} 0 & 0 & 0 \\ & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$B_{[2,2]'}B_{[4]'} = 4B_{[2,2]'} + B_{[1,3]} + B_{[1,1,1,1]}$$

$$4. B_{[4]}B_{[2]}.$$

$$\begin{pmatrix} 0 & 2 \\ & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ & 1 \\ 0 & 1 \end{pmatrix}$$

$$B_{[4]}B_{[2]} = 2B_{[2,2]} + B_{[2,1,1]'}$$

$$5. B_{[2]}B_{[2]}.$$

$$\begin{pmatrix} 2 & 0 \\ & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 0 \\ 1 & 1 \end{pmatrix}$$

$$B_{[2]}B_{[2]} = 2B_{[2]} + B_{[1,1]} + B_{[2,2]} + B_{[2,2]'} + 2B_{[1,1,1,1]}$$

6. $B_{[1,1]}B_{[2]}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ B_{[1,1]}B_{[2]} = 4B_{[1,1]} + 2B_{[1,1,2]} + 4B_{[1,1,1,1]}$$

Remark Note, in particular, that these examples illustrate the various conditions given in Theorem 1. Examples 1 and 2 illustrate conditions 1 and 2 respectively, and the influence of $\mathcal{B}(\mathbf{t}) = 0$ on possible filled templates belonging to $Z(p, q)$. Example 3 illustrates condition 3, and examples 4, 5 and 6 illustrate condition 4. More specifically, examples 4, 5 and 6 illustrate respectively conditions 4a, 4b when $r_1(\mathbf{t}) \geq 2$, and 4b when $r_1(\mathbf{t}) = 1$.

Corollary 1 $\mathcal{I} = \langle B_q | q \in C_1 \cup C_n \cup C'_n \rangle$ is an ideal.

PROOF Let B_p be a basis element of ΣD_n , and $B_q \in \mathcal{I}$. From our matrix interpretation it follows that any filled template, T , in $Z(p, q)$ or $Z(q, p)$ will be such that $z_{00} = 0$. Therefore $r(T) \models n$, that is $B_p B_q, B_q B_p \in \mathcal{I}$. The corollary follows immediately by linearity. ■

Moreover, we have the following.

Theorem 2 Let B_n be the Coxeter group of type B , whose Dynkin diagram is on n vertices, and let ΣB_n be its associated descent algebra. Then

$$\Sigma B_{n-2} \cong \Sigma D_n / \mathcal{I}$$

PROOF For clarity, for $q \in C_{<n}$, let B_q^D be a basis element of ΣD_n , and let B_q^B be a basis element of ΣB_{n-2} .

Note that the set $\{B_q^D\}_{q \in C_{<n}}$ is a basis for $\Sigma D_n / \mathcal{I}$. Hence, let $p \models m_1$, $q \models m_2$, $m_1, m_2 \leq n - 2$.

By Theorem 1, it follows that in $\Sigma D_n / \mathcal{I}$, the only non-zero term in the product $B_p^D B_q^D$ are those corresponding to filled templates in $Z(p, q)$ with $z_{00} \geq 2$. We denote this set of filled templates by $I(p, q)$. Note that if we subtract 2 from the z_{00} of any filled template, $T \in I(p, q)$, the reading word, row sum, and column sum of T are unaffected. Moreover, if this is performed on all $T \in I(p, q)$ the resulting filled templates are precisely those that arise if we calculate the

product $B_p^B B_q^B$ in ΣB_{n-2} ([2]). Since this argument is reversible, the result follows. ■

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